

SEMISTABLE HIGGS BUNDLES OVER COMPACT GAUDUCHON MANIFOLDS

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ABSTRACT. In this paper, we consider the existence of approximate Hermitian-Einstein structure and the semi-stability on Higgs bundles over compact Gauduchon manifolds. By using the continuity method, we show that they are equivalent.

1. INTRODUCTION

Let X be an n -dimensional compact complex manifold and g be a Hermitian metric with associated Kähler form ω . g is called to be Gauduchon if ω satisfies $\partial\bar{\partial}\omega^{n-1} = 0$. It has been proved by Gauduchon that if X is compact, there exists a Gauduchon metric ([13]) in the conformal class of every Hermitian metric g . In the following, we assume ω is Gauduchon.

Let (L, h) be a Hermitian line bundle over X . The ω -degree of L is defined by

$$\deg_{\omega}(L) := \int_X c_1(L, A_h) \wedge \frac{\omega^{n-1}}{(n-1)!},$$

where $c_1(L, A_h)$ is the first Chern form of L associated with the induced Chern connection A_h . Since $\partial\bar{\partial}\omega^{n-1} = 0$, $\deg_{\omega}(L)$ is well defined and independent of the choice of metric h ([24, p. 34-35]). Now given a rank s coherent analytic sheaf \mathcal{F} , we consider the determinant line bundle $\det \mathcal{F} = (\wedge^s \mathcal{F})^{**}$. Define the ω -degree of \mathcal{F} by

$$\deg_{\omega}(\mathcal{F}) := \deg_{\omega}(\det \mathcal{F}).$$

If \mathcal{F} is non-trivial and torsion free, the ω -slope of \mathcal{F} is defined by

$$\mu_{\omega}(\mathcal{F}) = \frac{\deg_{\omega}(\mathcal{F})}{\text{rank}(\mathcal{F})}.$$

Let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle over X . We say E is ω -stable (ω -semi-stable) in the sense of Mumford-Takemoto if for every proper coherent sub-sheaf $\mathcal{F} \hookrightarrow E$, there holds

$$\mu_{\omega}(\mathcal{F}) < \mu_{\omega}(E) \quad (\mu_{\omega}(\mathcal{F}) \leq \mu_{\omega}(E)).$$

A Hermitian metric H on E is said to be ω -Hermitian-Einstein if the Chern curvature F_H satisfies the Einstein condition

$$\sqrt{-1}\Lambda_{\omega}F_H = \lambda \cdot \text{Id}_E,$$

where $\lambda = \frac{2\pi\mu_{\omega}(E)}{\text{Vol}(X)}$. When the Kähler form is understood, we omit the subscript ω in the above definitions.

The Donaldson-Uhlenbeck-Yau theorem states that holomorphic vector bundles admit Hermitian-Einstein metrics if they are stable. It was proved by Narasimhan and Seshadri in [26] for compact

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Riemann surface case, by Donaldson in [10, 11] for algebraic manifolds and by Uhlenbeck and Yau in [28, 29] for general compact Kähler manifolds. The inverse problem that a holomorphic bundle admitting such a metric must be poly-stable(i.e. a direct sum of stable bundles with the same slope) was solved by Kobayashi [17] and Lübke [23] independently. Actually, this is the well-known Hitchin-Kobayashi correspondence for holomorphic vector bundles over compact Kähler manifolds. This correspondence is also valid for compact Gauduchon manifolds [8, 21, 24]. There are many other interesting generalized Hitchin-Kobayashi correspondences (see the references [1–5, 12, 14, 16, 19, 20, 27] for details).

A Higgs vector bundle $(E, \bar{\partial}_E, \phi)$ over X is a holomorphic vector bundle $(E, \bar{\partial}_E)$ together with a Higgs field $\phi \in \Omega_X^{1,0}(\text{End}(E))$ satisfying $\bar{\partial}_E \phi = 0$ and $\phi \wedge \phi = 0$. Higgs bundle was introduced by Hitchin [14] in his study of self dual equations on a Riemann surface, and studied by Simpson [27] in his work on nonabelian Hodge theory. It has a rich structure and plays an important role in many areas including gauge theory, Kähler geometry and hyperkähler geometry, group representations and non-abelian Hodge theory. A Higgs bundle $(E, \bar{\partial}_E, \phi)$ is stable(resp. semi-stable) if $\mu(\mathcal{F}) < \mu(E)$ (resp. $\mu(\mathcal{F}) \leq \mu(E)$) for every proper ϕ -invariant coherent subsheaf \mathcal{F} of E .

Given a Hermitian metric H on a Higgs bundle, we consider the Hitchin-Simpson connection ([27])

$$D_{H, \bar{\partial}_E, \phi} = D_{H, \bar{\partial}_E} + \phi + \phi^{*H},$$

where $D_{H, \bar{\partial}_E}$ is the Chern connection, and ϕ^{*H} is the adjoint of ϕ with respect to the metric H . The curvature of this connection is

$$F_{H, \bar{\partial}_E, \phi} = F_H + [\phi, \phi^{*H}] + \partial_H \phi + \bar{\partial}_E \phi^{*H},$$

where F_H is the curvature of the Chern connection $D_{H, \bar{\partial}_E}$. A Hermitian metric H on Higgs bundle $(E, \bar{\partial}_E, \phi)$ is said to be Hermitian-Einstein if the curvature $F_{H, \bar{\partial}_E, \phi}$ satisfies

$$\sqrt{-1}\Lambda_\omega F_{H, \bar{\partial}_E, \phi} = \sqrt{-1}\Lambda_\omega(F_H + [\phi, \phi^{*H}]) = \lambda \text{Id}_E.$$

Hitchin [14] and Simpson [27] proved that a Higgs bundle is poly-stable if and only if it admits a Hermitian-Einstein structure. This is a Higgs bundle version of the classical Hitchin-Kobayashi correspondence.

A Higgs bundle is said to be admitting an approximate Hermitian-Einstein structure, if for $\forall \varepsilon > 0$, there exists a Hermitian metric H_ε such that

$$\max_X | \sqrt{-1}\Lambda_\omega(F_{H_\varepsilon} + [\phi, \phi^{*H_\varepsilon}]) - \lambda \cdot \text{Id}_E |_{H_\varepsilon} < \varepsilon.$$

Kobayashi([18]) introduced this notion in a holomorphic vector bundle (i.e. $\phi = 0$). He proved that over a compact Kähler manifold, a holomorphic vector bundle admitting such a structure must be semi-stable. In [7], Bruzzo and Graña Otero generalized the above result to Higgs bundles. When X is projective, Kobayashi [18] solved the inverse part that a semi-stable holomorphic vector bundle must admit an approximate Hermitian-Einstein structure and conjectured that this should be true for general Kähler manifolds. This was confirmed in [9, 15, 22].

In this paper, we are interested in the existence of approximate Hermitian-Einstein structures on Higgs bundles over compact Gauduchon manifolds. In fact, we prove that:

Theorem 1.1. *Let (X, ω) be an n -dimensional compact Gauduchon manifold and $(E, \bar{\partial}_E, \phi)$ be a rank r Higgs bundle over X . Then $(E, \bar{\partial}_E, \phi)$ is semi-stable if and only if it admits an approximate Hermitian-Einstein structure.*

Now we give an overview of our proof. The difficult part of Theorem 1.1 is to prove the existence of approximate Hermitian-Einstein structure. In the Kähler case, by using the Donaldson heat flow, Li and Zhang ([22]) showed that the semi-stability implies admitting an approximate Hermitian-Einstein structure. Their proof relies on the properties of the Donaldson functional. However, the Donaldson functional is not well-defined if ω is only Gauduchon. So Li and Zhang's argument can not be generalized to Gauduchon manifold case directly. In this paper, we use the continuity method to prove the existence. Fixed a proper background Hermitian metric H_0 on E , we consider the following perturbed equation

$$(1.1) \quad L_\varepsilon(f) := \mathcal{K}_H - \lambda \text{Id}_E + \varepsilon \log f = 0, \quad \varepsilon \in (0, 1],$$

where $\mathcal{K}_H = \sqrt{-1} \Lambda_\omega F_{H, \bar{\partial}_E, \phi} = K_H + \sqrt{-1} \Lambda_\omega [\phi, \phi^{*H}]$ and $f = H_0^{-1} \cdot H$. It is obvious that f and $\log f$ are self adjoint with respect to H_0 and H . By the results of Lübke and Teleman in [24, 25], (1.1) is solvable for $\forall \varepsilon \in (0, 1]$. Under the assumption of semi-stability, we can show that

$$(1.2) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \max_X |\log f_\varepsilon|_{H_0} = 0.$$

This implies that $\max_X |\mathcal{K}_{H_\varepsilon} - \lambda \cdot \text{Id}_E|_{H_\varepsilon}$ converges to zero as $\varepsilon \rightarrow 0$ (see Theorem 3.2 for details).

This article is organised as below. In Sect.2, we present some basic estimates for the perturbed equation (1.1). In Sect.3, we prove Theorem 1.1 in detail.

2. PRELIMINARY

Let $(E, \bar{\partial}_E, \phi)$ be a Higgs bundle over X and H be a Hermitian metric on E . Set

$$\text{Herm}(E, H) = \{\eta \in \text{End}(E) \mid \eta^{*H} = \eta\}$$

and

$$\text{Herm}^+(E, H) = \{\rho \in \text{Herm}(E, H) \mid H\rho \text{ is positive definite}\}.$$

Suppose $f \in \text{Herm}^+(E, H_0)$ is a solution of the equation (1.1) with the background metric H_0 for some $\varepsilon \in (0, 1]$. Substituting

$$\mathcal{K}_H = \mathcal{K}_{H_0} + \sqrt{-1} \Lambda_\omega (\bar{\partial}(f^{-1} \circ \partial_{H_0} f) + [\phi, \phi^{*H} - \phi^{*H_0}])$$

into (1.1), we obtain

$$(2.1) \quad L_\varepsilon(f) = \mathcal{K}_{H_0} - \lambda \text{Id}_E + \sqrt{-1} \Lambda_\omega (\bar{\partial}(f^{-1} \circ \partial_{H_0} f) + [\phi, \phi^{*H} - \phi^{*H_0}]) + \varepsilon \log f = 0.$$

Furthermore, by an appropriate conformal change, we can assume that H_0 satisfies

$$\text{tr}(\mathcal{K}_{H_0} - \lambda \text{Id}_E) = 0.$$

In fact, let $H_0 = e^\varphi H'_0$, where H'_0 is an arbitrary metric and φ is a smooth function satisfying

$$(2.2) \quad \sqrt{-1} \Lambda_\omega \bar{\partial} \partial(\varphi) = -\frac{1}{r} \text{tr}(\mathcal{K}_{H'_0} - \lambda \cdot \text{Id}_E).$$

Since $\int_X \text{tr}(\mathcal{K}_{H'_0} - \lambda \text{Id}_E) \omega^n = 0$, equation (2.2) is solvable.

For simplicity, we set $\Phi(H, \phi) = \mathcal{K}_H - \lambda \cdot \text{Id}_E$. It is easily to check that $\Phi(H, \phi)^{*H} = \Phi(H, \phi)$. The following two lemmas are proved by Teleman and Lübke in [24]. Here we present the proofs just for readers' convenience.

Lemma 2.1. *Fix a background Hermitian metric H_0 satisfying $\text{tr}\Phi(H_0, \phi) = 0$. Then for any $f \in \text{Herm}^+(E, H_0)$ such that $L_\varepsilon(f) = 0$, it holds*

$$P(\text{tr} \log f) + \varepsilon \text{tr} \log f = 0,$$

where P is denoted by $P = \sqrt{-1}\Lambda_\omega \bar{\partial}\partial$. Furthermore, we have $\det f = 1$.

Proof. By $\partial \log \det f = \text{Tr}(f^{-1}\partial f)$ and $\log \det f = \text{tr} \log f$, we have

$$\begin{aligned} \text{Tr} \sqrt{-1} \wedge_\omega (\bar{\partial}(f^{-1}\partial_{H_0} f)) &= \sqrt{-1} \wedge_\omega \bar{\partial} \text{Tr}(f^{-1}\partial f) \\ (2.3) \quad &= \sqrt{-1} \wedge_\omega \bar{\partial} \partial \log \det f \\ &= \sqrt{-1} \wedge_\omega \bar{\partial} \partial \text{tr} \log f. \end{aligned}$$

Then combining (2.3) with $\text{Tr} \sqrt{-1} \wedge_\omega [\phi, f^{-1}\phi^{*H_0}f - \phi^{H_0}] = 0$, we conclude that

$$\begin{aligned} 0 &= \text{tr} L_\varepsilon(f) \\ &= \text{tr} \Phi(H_0, \phi) + \text{tr} \sqrt{-1} \wedge_\omega (\bar{\partial}(f^{-1}\partial_{H_0} f)) + \varepsilon \text{tr} \log f \\ &= P(\text{tr} \log f) + \varepsilon \text{tr} \log f. \end{aligned}$$

Furthermore, by the maximum principle, we have $\text{tr} \log f = 0$ and $\det f = 1$. \square

Lemma 2.2. *If $f \in \text{Herm}^+(E, H_0)$ satisfies $L_\varepsilon(f) = 0$ for some $\varepsilon > 0$, then there holds that*

- (i) $\frac{1}{2}P(|\log f|_{H_0}^2) + \varepsilon |\log f|_{H_0}^2 \leq |\Phi(H_0, \phi)|_{H_0} |\log f|_{H_0}$;
- (ii) $m = \max_X |\log f|_{H_0} \leq \frac{1}{\varepsilon} \cdot \max_X |\Phi(H_0, \phi)|_{H_0}$;
- (iii) $m \leq C \cdot (\|\log f\|_{L^2} + \max_X |\Phi(H_0, \phi)|_{H_0})$, where C only depends on g and X .

Proof. (i) Taking the point-wise inner product with $\log f$ respect to H_0 of both sides of (2.1), we have

$$\begin{aligned} (2.4) \quad &\langle \sqrt{-1}\Lambda_\omega \bar{\partial}(f^{-1} \circ \partial_{H_0} f), \log f \rangle_{H_0} + \langle \sqrt{-1}\Lambda_\omega [\phi, \phi^{*H} - \phi^{*H_0}], \log f \rangle_{H_0} \\ &+ \varepsilon |\log f|_{H_0}^2 = -\langle \Phi(H_0, \phi), \log f \rangle_{H_0}. \end{aligned}$$

Set $A = \langle \sqrt{-1}\Lambda_\omega \bar{\partial}(f^{-1} \circ \partial_{H_0} f), \log f \rangle_{H_0}$ and $B = \langle \sqrt{-1}\Lambda_\omega [\phi, \phi^{*H} - \phi^{*H_0}], \log f \rangle_{H_0}$. From the result in [24, p. 74], we have

$$(2.5) \quad P(|\log f|_{H_0}^2) \leq 2A.$$

Now we estimate B . Let $H(t) = H_0 e^{ts}$, $t \in [0, 1]$ be a curve in $\text{Herm}^+(E)$ connecting H_0 and $H_0 f$, where $s = \log f$. Set $\zeta(t) = H_0 (\sqrt{-1}\Lambda_\omega [\phi, e^{-ts}\phi^{*H_0}e^{ts}], s)$. The t -derivative of $\zeta(t)$ is

$$\frac{d}{dt}\zeta(t) = \langle \sqrt{-1}\Lambda_\omega [\phi, -se^{-ts}\phi^{*H_0}e^{ts} + e^{-ts}\phi^{*H_0}e^{ts}s], s \rangle_{H_0} = | [s, e^{\frac{ts}{2}} \phi e^{-\frac{ts}{2}}] |_{H_0}^2 \geq 0.$$

This implies

$$(2.6) \quad B = \zeta(1) \geq \zeta(0) = 0.$$

By (2.4-2.6), we have

$$\frac{1}{2}P(|\log f|_{H_0}^2) + \varepsilon |\log f|_{H_0}^2 \leq -\langle \Phi(H_0, \phi), \log f \rangle_{H_0} \leq |\Phi(H_0, \phi)|_{H_0} |\log f|_{H_0}.$$

(ii) Assuming $|\log f|_{H_0}^2$ attains its maximum at $p \in X$, we have

$$0 \leq \frac{1}{2}P(|\log f|^2)(p) \leq (|\Phi(H_0, \phi)|_{H_0}(p) - \varepsilon |\log f|_{H_0}(p)) |\log f|_{H_0}(p).$$

Then it follows that

$$\max_X |\log f|_{H_0} |\log f|_{H_0}(p) \leq \frac{1}{\varepsilon} |\Phi(H_0, \phi)|_{H_0}(p) \leq \frac{1}{\varepsilon} \max_X |\Phi(H_0, \phi)|_{H_0}.$$

(iii) From (i), we get

$$P(|\log f|_{H_0}^2) \leq |\Phi(H_0, \phi)|_{H_0}^2 + |\log f|_{H_0}^2 \leq \max_X |\Phi(H_0, \phi)|_{H_0}^2 + |\log f|_{H_0}^2.$$

Then by Moser's iteration, there exist a constant $C > 0$ depending on g and X such that

$$m \leq C \cdot (\|\log f\|_{L^2} + \max_X |\Phi(H_0, \phi)|_{H_0}).$$

□

3. PROOF OF THEOREM 1.1

Before we give the detailed proof, we recall some notation. Given $\eta \in \text{Herm}(E, H)$, from [24, p. 237], we can choose an open dense subset $W \subseteq X$ satisfying at each $x \in W$ there exist an open neighbourhood U of x , a local unitary basis $\{e_a\}_{a=1}^r$ respect to H and functions $\{\lambda_a \in C^\infty(U, R)\}_{a=1}^r$ such that

$$\eta(y) = \sum_{a=1}^r \lambda_a(y) \cdot e_a(y) \otimes e^a(y)$$

for all $y \in U$, where $\{e^a\}_{a=1}^r$ denotes the dual basis of E^* . Let $\varphi \in C^\infty(R, R)$, $\Psi \in C^\infty(R \times R, R)$ and $A = \sum_{a,b=1}^r A_a^b e^a \otimes e_b \in \text{End}(E)$. We denote $\varphi(\eta)$ and $\Psi(\eta)(A)$ by

$$(3.1) \quad \varphi(\eta)(y) = \sum_{a=1}^r \varphi(\lambda_a) e_a \otimes e^a$$

and

$$(3.2) \quad \Psi(\eta)(A)(y) = \Psi(\lambda_a, \lambda_b) A_a^b e^a \otimes e_b.$$

Proposition 3.1. *If $f \in \text{Herm}^+(E, H_0)$ solves (2.1) for some ε , then there holds*

$$(3.3) \quad \int_X \text{tr}(\Phi(H_0, \phi)s) \frac{\omega^n}{n!} + \int_X \langle \Psi(s)(\mathcal{D}''s), \mathcal{D}''s \rangle_{H_0} \frac{\omega^n}{n!} = -\varepsilon \|s\|_{L^2}^2,$$

where $s = \log f$, $\mathcal{D}'' = \bar{\partial}_E + \phi$ and

$$\Psi(x, y) = \begin{cases} \frac{e^{y-x}-1}{y-x}, & x \neq y; \\ 1, & x = y. \end{cases}$$

Proof. First, (2.1) gives

$$(3.4) \quad \begin{aligned} & \int_X \text{tr}(\Phi(H_0, \phi)s) \frac{\omega^n}{n!} + \int_X \langle \sqrt{-1} \Lambda_\omega \bar{\partial}(f^{-1} \partial_{H_0} f), s \rangle_{H_0} \frac{\omega^n}{n!} \\ & + \int_X \langle \sqrt{-1} \Lambda_\omega [\phi, \phi^{*H} - \phi^{*H_0}], s \rangle_{H_0} \frac{\omega^n}{n!} + \varepsilon \|s\|_{L^2}^2 = 0, \end{aligned}$$

where $H = H_0 f$. Then comparing (3.4) with (3.3), it is sufficient to show

$$(3.5) \quad \int_X \langle \sqrt{-1} \Lambda_\omega \bar{\partial}(f^{-1} \partial_{H_0} f) + [\phi, \phi^{*H} - \phi^{*H_0}], s \rangle_{H_0} \frac{\omega^n}{n!} = \int_X \langle \Psi(s)(\mathcal{D}''s), \mathcal{D}''s \rangle_{H_0} \frac{\omega^n}{n!}.$$

We will divide the proof of (3.5) into the following two steps.

Step 1 We show that

$$(3.6) \quad \int_X \langle \sqrt{-1} \Lambda_\omega \bar{\partial}(f^{-1} \partial_{H_0} f) + [\phi, \phi^{*H} - \phi^{*H_0}], s \rangle_{H_0} \frac{\omega^n}{n!} = \int_X \text{Tr} \sqrt{-1} \Lambda_\omega \{f^{-1} \mathcal{D}' f \wedge \mathcal{D}'' s\} \frac{\omega^n}{n!},$$

where $\mathcal{D}' = \partial_{H_0} + \phi^{*H_0}$.

By using Stokes formula, we have

$$(3.7) \quad \begin{aligned} & \int_X \langle \sqrt{-1} \Lambda_\omega \bar{\partial}(f^{-1} \partial_{H_0} f), s \rangle_{H_0} \frac{\omega^n}{n!} \\ &= \int_X \bar{\partial} \left(\text{Tr} \{ \sqrt{-1} f^{-1} (\partial_{H_0} f) s \} \frac{\omega^{n-1}}{(n-1)!} \right) + \int_X \text{Tr} \{ \sqrt{-1} f^{-1} \partial_{H_0} f \bar{\partial} s \} \frac{\omega^{n-1}}{(n-1)!} \\ & \quad + \int_X \text{Tr} \{ \sqrt{-1} f^{-1} (\partial_{H_0} f) s \} \bar{\partial} \frac{\omega^{n-1}}{(n-1)!} \\ &= \int_X \text{Tr} \{ \sqrt{-1} f^{-1} \partial_{H_0} f \bar{\partial} s \} \frac{\omega^{n-1}}{(n-1)!} + \int_X \text{Tr} \{ \sqrt{-1} f^{-1} (\partial_{H_0} f) s \} \bar{\partial} \frac{\omega^{n-1}}{(n-1)!}. \end{aligned}$$

Since $sf = fs$, it follows that

$$(3.8) \quad \begin{aligned} \text{Tr} (f^{-1} (\partial_{H_0} f) s) &= \text{Tr} \left(f^{-1} (\partial_{H_0} \sum_{k=1}^{+\infty} \frac{s^k}{k!}) s \right) = \text{Tr} \left(f^{-1} \sum_{k=1}^{+\infty} \sum_{j=0}^{k-1} \frac{s^j (\partial_{H_0} s) s^{k-1-j}}{k!} s \right) \\ &= \text{Tr} \left(f^{-1} \sum_{k=1}^{+\infty} \sum_{j=0}^{k-1} \frac{s^k \partial_{H_0} s}{k!} \right) = \text{Tr} \left(f^{-1} \sum_{k=1}^{+\infty} \frac{s^k \partial_{H_0} s}{(k-1)!} \right) \\ &= \text{Tr} \left(f^{-1} \sum_{k=1}^{+\infty} \frac{s^{k-1}}{(k-1)!} s \partial_{H_0} s \right) = \text{Tr} (s \partial_{H_0} s). \end{aligned}$$

(3.8) together with $\partial \bar{\partial} \omega^{n-1} = 0$ gives

$$(3.9) \quad \int_X \text{Tr} (\sqrt{-1} f^{-1} (\partial_{H_0} f) s) \bar{\partial} \frac{\omega^{n-1}}{(n-1)!} = \int_X \frac{1}{2} \partial \text{Tr} (\sqrt{-1} s^2) \bar{\partial} \frac{\omega^{n-1}}{(n-1)!} = 0.$$

From (3.7) and (3.9), we have

$$(3.10) \quad \int_X \langle \sqrt{-1} \Lambda_\omega \bar{\partial}(f^{-1} \partial_{H_0} f), s \rangle_{H_0} \frac{\omega^n}{n!} = \int_X \text{Tr} \{ \sqrt{-1} f^{-1} \partial_{H_0} f \bar{\partial} s \} \frac{\omega^{n-1}}{(n-1)!}.$$

Then noticing that $\text{tr}(AB) = (-1)^{pq} \text{tr}(BA)$, where A is an $\text{End}(E)$ valued p -form and B is an $\text{End}(E)$ valued q -form, there holds

$$(3.11) \quad \begin{aligned} & \int_X \text{Tr} \{ \sqrt{-1} \Lambda_\omega [\phi, f^{-1} \phi^{*H_0} f - \phi^{*H_0}] s \} \frac{\omega^n}{n!} \\ &= \int_X \sqrt{-1} \text{Tr} \{ \phi f^{-1} \phi^{*H_0} f s + f^{-1} \phi^{*H_0} f \phi s - (\phi \phi^{*H_0} s + \phi^{*H_0} \phi s) \} \frac{\omega^{n-1}}{(n-1)!} \\ &= \int_X \sqrt{-1} \text{Tr} \{ -f^{-1} \phi^{*H_0} f s \phi + f^{-1} \phi^{*H_0} f \phi s + \phi^{*H_0} s \phi - \phi^{*H_0} \phi s \} \frac{\omega^{n-1}}{(n-1)!} \\ &= \int_X \sqrt{-1} \text{Tr} \{ f^{-1} [\phi^{*H_0}, f] [\phi, s] \} \frac{\omega^{n-1}}{(n-1)!}. \end{aligned}$$

Therefore, we complete Step 1 by substituting (3.10) and (3.11) into the left hand side of (3.6).

Step 2 We show that

$$(3.12) \quad \text{Tr} \sqrt{-1} \Lambda_\omega \{f^{-1} \mathcal{D}' f \wedge \mathcal{D}'' s\} = \langle \Psi(s)(\mathcal{D}'' s), \mathcal{D}'' s \rangle_{H_0}$$

holds on X .

From [24, p. 237-238], there exists an open dense subset $W \subseteq X$ such that at each $x \in W$, one has

$$\mathcal{D}' f(x) = e^{\lambda_a} \partial \lambda_a e_a \otimes e^a + (e^{\lambda_b} - e^{\lambda_a})(A_b^a + \overline{\phi_b^a}) e_a \otimes e^b$$

and

$$\mathcal{D}'' s(x) = \overline{\partial} \lambda_a e_a \otimes e^a + (\lambda_b - \lambda_a) \left(-\overline{A_b^a} + \phi_b^a \right) e_a \otimes e^b,$$

where $\{e_a\}_{a=1}^r$ is a local unitary basis of E respect to H_0 and the $(1, 0)$ -forms A_a^b are defined by $\partial_{H_0} e_a = A_a^b e_b$.

It follows that at each $x \in W$,

$$\begin{aligned} & \text{Tr} \sqrt{-1} \Lambda_\omega \{f^{-1} \mathcal{D}' f \wedge \mathcal{D}'' s\} \\ &= \sum_{a=1}^r |\overline{\partial} \lambda_a|^2 + \sum_{a \neq b} (e^{\lambda_b - \lambda_a} - 1)(\lambda_a - \lambda_b) \sqrt{-1} \Lambda_\omega (A_b^a + \overline{\phi_b^a}) \wedge (-\overline{A_b^a} + \phi_b^a) \\ &= \sum_{a=1}^r |\overline{\partial} \lambda_a|^2 + \sum_{a \neq b} \frac{e^{\lambda_b - \lambda_a} - 1}{\lambda_b - \lambda_a} (\lambda_b - \lambda_a)^2 |-\overline{A_b^a} + \phi_b^a|^2 \\ &= \sum_{a,b} \Psi(\lambda_a, \lambda_b) |(\mathcal{D}'' s)_a^b|^2. \end{aligned}$$

We now turn to calculating the right hand side of (3.12). By the construction (3.2), we have

$$\begin{aligned} \Psi(s)(\mathcal{D}'' s) &= \overline{\partial} \lambda_a e_a \otimes e^a + \Psi(\lambda_a, \lambda_b)(\lambda_a - \lambda_b)(-\overline{A_b^a} + \phi_b^a) e_b \otimes e^a \\ &= \sum_{a=1}^r \overline{\partial} \lambda_a e_a \otimes e^a + \sum_{a \neq b} \frac{e^{\lambda_b - \lambda_a} - 1}{\lambda_b - \lambda_a} (\lambda_a - \lambda_b)(-\overline{A_b^a} + \phi_b^a) e_b \otimes e^a \end{aligned}$$

Then at each $x \in W$ there holds

$$\begin{aligned} \langle \Psi(s)(\mathcal{D}'' s), \mathcal{D}'' s \rangle_{H_0} &= \sum_{a=1}^r |\overline{\partial} \lambda_a|^2 + \sum_{a \neq b} (e^{\lambda_b - \lambda_a} - 1)(\lambda_b - \lambda_a) |-\overline{A_b^a} + \phi_b^a|^2 \\ &= \sum_{a,b} \Psi(\lambda_a, \lambda_b) |(\mathcal{D}'' s)_a^b|^2 \\ &= \text{Tr} \sqrt{-1} \Lambda_\omega \{f^{-1} \mathcal{D}' f \wedge \mathcal{D}'' s\}. \end{aligned}$$

This forces

$$(3.13) \quad \langle \Psi(s)(\mathcal{D}'' s), \mathcal{D}'' s \rangle_{H_0} = \text{Tr} \sqrt{-1} \Lambda_\omega \{f^{-1} \mathcal{D}' f \wedge \mathcal{D}'' s\}$$

holds on X . So, combining (3.6) with (3.13) we have (3.5). \square

Then, we prove the “only if” part of Theorem 1.1. In fact, we prove the following theorem

Theorem 3.2. *If Higgs bundle $(E, \overline{\partial}_E, \phi)$ is ω -semi-stable, then $\max_X |\Phi(H_\varepsilon, \phi)|_{H_\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$.*

Proof. Let $\{f_\varepsilon\}_{0 < \varepsilon \leq 1}$ be the solutions of equation (2.1) with background metric H_0 . Then there holds that

$$\|\log f_\varepsilon\|_{L^2}^2 = -\frac{1}{\varepsilon} \int_X \langle \Phi(H_\varepsilon, \phi), \log f \rangle_{H_\varepsilon} \frac{\omega^n}{n!}.$$

Case 1, $\exists C_1 > 0$ such that $\|\log f_\varepsilon\|_{L^2} < C_1 < +\infty$. From Lemma 2.2, we have

$$\max_X |\Phi(H_\varepsilon, \phi)|_{H_\varepsilon} = \varepsilon \cdot \max_X |\log f_\varepsilon|_{H_\varepsilon} < \varepsilon C \cdot (C_1 + \max_X |\Phi(H_0, \phi)|_{H_0}).$$

Then it follows that $\max_X |\Phi(H_\varepsilon, \phi)|_{H_\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Case 2, $\lim_{\varepsilon \rightarrow 0} \|\log f_\varepsilon\|_{L^2} \rightarrow \infty$.

Claim If $(E, \bar{\partial}_E, \phi)$ is semi-stable, there holds

$$(3.14) \quad \lim_{\varepsilon \rightarrow 0} \max_X |\Phi(H_\varepsilon, \phi)|_{H_\varepsilon} = \lim_{\varepsilon \rightarrow 0} \varepsilon \max_X |\log f_\varepsilon|_{H_\varepsilon} = 0.$$

We will follow Simpson's argument ([27, Proposition 5.3]) to show that if the claim does not hold, there exists a Higgs subsheaf contradicting the semi-stability.

If the claim does not hold, then there exist $\delta > 0$ and a subsequence $\varepsilon_i \rightarrow 0$, $i \rightarrow +\infty$, such that

$$\|\log f_{\varepsilon_i}\|_{L^2} \rightarrow +\infty$$

and

$$(3.15) \quad \max_X |\Phi(H_{\varepsilon_i}, \phi)|_{H_{\varepsilon_i}} = \varepsilon_i \max_X |\log f_{\varepsilon_i}|_{H_{\varepsilon_i}} \geq \delta.$$

Setting $s_{\varepsilon_i} = \log f_{\varepsilon_i}$, $l_i = \|s_{\varepsilon_i}\|_{L^2}$ and $u_{\varepsilon_i} = s_{\varepsilon_i}/l_i$, it follows that $\text{tr} u_{\varepsilon_i} = 0$ and $\|u_{\varepsilon_i}\|_{L^2} = 1$. Then combining (3.15) with Lemma 2.2 (iii), we have

$$(3.16) \quad l_i \geq \frac{\delta}{C\varepsilon_i} - \max_X |\Phi(H_0, \phi)|_{H_0}.$$

and

$$(3.17) \quad \max_X |u_{\varepsilon_i}| < \frac{C}{l_i} (l_i + \max_X |\Phi(H_0, \phi)|) < C_2 < +\infty.$$

Step 1 We show that $\|u_{\varepsilon_i}\|_{L^2_1}$ are uniformly bounded. Since $\|u_{\varepsilon_i}\|_{L^2} = 1$, we only need to prove $\|\mathcal{D}'' u_{\varepsilon_i}\|_{L^2}$ are uniformly bounded.

By (1.1) and Proposition 3.1, for each f_{ε_i} , there holds

$$(3.18) \quad \int_X \text{Tr}\{\Phi(H_0, \phi)u_{\varepsilon_i}\} \frac{\omega^n}{n!} + l_i \int_X \langle \Psi(l_i u_{\varepsilon_i})(\mathcal{D}'' u_{\varepsilon_i}), \mathcal{D}'' u_{\varepsilon_i} \rangle_{H_0} \frac{\omega^n}{n!} = -\varepsilon_i l_i$$

Substituting (3.16) into (3.18), we have

$$(3.19) \quad C^* + \int_X \text{Tr}\{\Phi(H_0, \phi)u_{\varepsilon_i}\} + \langle l_i \Psi(l_i u_{\varepsilon_i})(\mathcal{D}'' u_{\varepsilon_i}), \mathcal{D}'' u_{\varepsilon_i} \rangle_{H_0} \frac{\omega^n}{n!} \leq \varepsilon_i \max_X |\Phi(H_0, \phi)|_{H_0},$$

where $C^* = \frac{\delta}{C}$.

Consider the function

$$(3.20) \quad l\Psi(lx, ly) = \begin{cases} l, & x = y; \\ \frac{e^{l(y-x)} - 1}{y-x}, & x \neq y. \end{cases}$$

From (3.17), we may assume that $(x, y) \in [-C_2, C_2] \times [-C_2, C_2]$. It is easy to check that

$$(3.21) \quad l\Psi(lx, ly) \longrightarrow \begin{cases} (x-y)^{-1}, & x > y; \\ +\infty, & x \leq y, \end{cases}$$

increases monotonically as $l \rightarrow +\infty$. Let $\zeta \in C^\infty(R \times R, R^+)$ satisfying $\zeta(x, y) < (x - y)^{-1}$ whenever $x > y$. From (3.19), (3.21) and the arguments in Lemma 5.4([27]), we have

$$(3.22) \quad C^* + \int_X \text{tr}\{u_{\varepsilon_i} \Phi(H_0, \phi)\} + \langle \zeta(u_{\varepsilon_i}) \mathcal{D}'' u_{\varepsilon_i}, \mathcal{D}'' u_{\varepsilon_i} \rangle_{H_0} \frac{\omega^n}{n!} \leq \varepsilon_i \max_X |\Phi(H_0, \phi)|_{H_0},$$

when $i \gg 0$. Particularly, we take $\zeta(x, y) = \frac{1}{2C_2}$. It is obvious that when $(x, y) \in [-C_2, C_2] \times [-C_2, C_2]$ and $x > y$, $\frac{1}{3C_2} < \frac{1}{x-y}$. This implies that

$$C^* + \int_X \text{tr}\{u_{\varepsilon_i} \Phi(H_0, \phi)\} + \frac{1}{3C_2} |\mathcal{D}'' u_{\varepsilon_i}|_{H_0}^2 \frac{\omega^n}{n!} \leq \varepsilon_i \max_X |\Phi(H_0, \phi)|_{H_0},$$

when $i \gg 0$. Then we have

$$\int_X |\mathcal{D}'' u_{\varepsilon_i}|_{H_0}^2 \frac{\omega^n}{n!} \leq 3C_2 \max_X |\Phi(H_0, \phi)|_{H_0} (\text{Vol}(X)^{\frac{1}{2}} + 1).$$

Thus, u_{ε_i} are bounded in L_1^2 . We can choose subsequence $\{u_{\varepsilon_{i_j}}\}$ such that $u_{\varepsilon_{i_j}} \rightharpoonup u_\infty$ weakly in L_1^2 . We still write it $\{u_{\varepsilon_i}\}_{i=1}^\infty$ for simplicity. Noting that $L_1^2 \hookrightarrow L^2$, we have

$$1 = \int_X |u_{\varepsilon_i}|_{H_0}^2 \rightarrow \int_X |u_\infty|_{H_0}^2.$$

This indicates that $\|u_\infty\|_{L^2} = 1$ and u_∞ is nontrivial.

So by (3.22) and the same discussion in Lemma 5.4 ([27]), there holds

$$(3.23) \quad C^* + \int_X \text{tr}\{u_\infty \Phi(H_0, \phi)\} + \langle \zeta(u_\infty) \mathcal{D}'' u_\infty, \mathcal{D}'' u_\infty \rangle_{H_0} \frac{\omega^n}{n!} \leq 0.$$

Step 2 By using Uhlenbeck and Yau's trick in [28] to construct a Higgs sub-sheaf which contradicts the semi-stability of E .

From (3.23) and the technique in Lemma 5.5 in ([27]), we have the eigenvalues of u_∞ are constant almost everywhere. Let $\mu_1 < \mu_2 < \dots < \mu_l$ be the distinct eigenvalues of u_∞ . The facts that $\text{tr}(u_\infty) = \text{tr}(u_{\varepsilon_i}) = 0$ and $\|u_\infty\|_{L^2} = 1$ force $2 \leq l \leq r$. For each μ_α ($1 \leq \alpha \leq l-1$), we construct a function $P_\alpha : R \rightarrow R$ such that

$$P_\alpha = \begin{cases} 1, & x \leq \mu_\alpha \\ 0, & x \geq \mu_{\alpha+1}. \end{cases}$$

Setting $\pi_\alpha = P_\alpha(u_\infty)$, from [27, p. 887], we have

- (i) $\pi_\alpha \in L_1^2$;
- (ii) $\pi_\alpha^2 = \pi_\alpha = \pi_\alpha^{*H_0}$;
- (iii) $(\text{Id} - \pi_\alpha) \bar{\partial} \pi_\alpha = 0$;
- (iv) $(\text{Id} - \pi_\alpha)[\phi, \pi_\alpha] = 0$.

By Uhlenbeck and Yau's regularity statement of L_1^2 -subbundle ([28]), $\{\pi_\alpha\}_{\alpha=1}^{l-1}$ determine $l-1$ Higgs sub-sheaves of E . Set $E_\alpha = \pi_\alpha(E)$. Since $\text{tr} u_\infty = 0$ and $u_\infty = \mu_l \text{Id} - \sum_{\alpha=1}^{l-1} (\mu_{\alpha+1} - \mu_\alpha) \pi_\alpha$, there holds

$$(3.24) \quad \mu_l \text{rank} E = \sum_{\alpha=1}^{l-1} (\mu_{\alpha+1} - \mu_\alpha) \text{rank} E_\alpha,$$

Construct

$$\nu = \mu_l \deg(E) - \sum_{\alpha=1}^{l-1} (\mu_{\alpha+1} - \mu_\alpha) \deg(E_\alpha).$$

From one hand, substituting (3.24) into ν ,

$$(3.25) \quad \nu = \sum_{\alpha=1}^{l-1} (\mu_{\alpha+1} - \mu_\alpha) \text{rank} E_\alpha \left(\frac{\deg(E)}{\text{rank} E} - \frac{\deg(E_\alpha)}{\text{rank} E_\alpha} \right)$$

From the other hand, substituting the Chern-Weil formula (Prop. 2.3 in [6])

$$\deg(E_\alpha) = \int_X \text{Tr}(\pi_\alpha \mathcal{K}_{H_0}) - |\mathcal{D}'' \pi_\alpha|^2 \frac{\omega^n}{n!}$$

into ν ,

$$\begin{aligned} \nu &= \mu_l \int_X \text{Tr}(\mathcal{K}_{H_0}) - \sum_{\alpha=1}^{l-1} (\mu_{\alpha+1} - \mu_\alpha) \left\{ \int_X \text{Tr}(\pi_\alpha \mathcal{K}_{H_0}) - \int_X |\mathcal{D}'' \pi_\alpha|^2_{H_0} \right\} \\ &= \int_X \text{Tr} \left((\mu_l \text{Id} - \sum_{\alpha=1}^{l-1} (\mu_{\alpha+1} - \mu_\alpha) \pi_\alpha) \mathcal{K}_{H_0} + \sum_{\alpha=1}^{l-1} (\mu_{\alpha+1} - \mu_\alpha) \int_X |\mathcal{D}'' \pi_\alpha|^2 \right) \\ &= \int_X \text{Tr}(u_\infty \mathcal{K}_{H_0}) + \langle \sum_{\alpha=1}^{l-1} (\mu_{\alpha+1} - \mu_\alpha) (dP_\alpha)^2(u_\infty) (\mathcal{D}'' u_\infty), \mathcal{D}'' u_\infty \rangle_{H_0}, \end{aligned}$$

where the function $dP_\alpha : R \times R \rightarrow R$ is defined by

$$dP_\alpha(x, y) = \begin{cases} \frac{P_\alpha(x) - P_\alpha(y)}{x - y}, & x \neq y; \\ P'_\alpha(x), & x = y. \end{cases}$$

From simple calculation, we have if $\mu_\beta \neq \mu_\gamma$

$$(3.26) \quad \sum_{\alpha=1}^{l-1} (\mu_{\alpha+1} - \mu_\alpha) (dP_\alpha)^2(\mu_\beta, \mu_\gamma) = |\mu_\beta - \mu_\gamma|^{-1}.$$

Since $\text{tr} u_\infty = 0$, so by (3.23) and the same arguments in [22, p. 793-794] there holds

$$(3.27) \quad \nu = \int_X \text{Tr}(u_\infty \Phi(H_0, \phi)) + \langle \sum_{\alpha=1}^{l-1} (\mu_{\alpha+1} - \mu_\alpha) (dP_\alpha)^2(u_\infty) (\mathcal{D}'' u_\infty), \mathcal{D}'' u_\infty \rangle_{H_0} < -C^*.$$

Combining (3.25) with (3.27), we have

$$\sum_{\alpha=1}^{l-1} (\mu_{\alpha+1} - \mu_\alpha) \text{rank} E_\alpha \left(\frac{\deg(E)}{\text{rank} E} - \frac{\deg(E_\alpha)}{\text{rank} E_\alpha} \right) < 0.$$

This indicates there must exist a term $(\mu(E) - \mu(E_{\alpha_0})) < 0$, which contradicts the semi-stability of E . \square

Finally, we prove the “if ” part of the Theorem 1.1.

Theorem 3.3. *Let (X, ω) be an n -dimensional compact Gauduchon manifold and $(E, \bar{\partial}_E, \phi)$ be a Higgs bundle over X . If $(E, \bar{\partial}_E, \phi)$ admits an approximate Hermitian-Einstein manifold, then $(E, \bar{\partial}_E, \phi)$ is ω -semi-stable.*

Firstly, following the techniques of Kobayashi [17] and Bruzzo-Graña Otero's [7], we prove a Higgs version vanishing theorem.

Proposition 3.4. *Let (X, ω) be an n -dimensional Hermitian manifold with Gauduchon metric ω and $(E, \bar{\partial}_E, \phi)$ be a Higgs bundle over X . Assume that E admits an approximate Hermitian Einstein structure. If $\deg E < 0$, then E has no nonzero ϕ -invariant sections of E .*

Proof. Let H be a Hermitian metric over E and s be a ϕ -invariant holomorphic section of E . From simple calculation, one has

$$H(s, \sqrt{-1}\Lambda_\omega[\phi, \phi^{*H}]s) \geq 0.$$

Then we have the Weitzenböck formula

$$\begin{aligned} \sqrt{-1}\Lambda_\omega \partial \bar{\partial} H(s, s) &= |\partial_H s|^2 + H(s, -\sqrt{-1}\Lambda_\omega \bar{\partial} \partial_H s) \\ (3.28) \quad &= |\partial_H s|^2 + H(s, -\mathcal{K}_H(s)) + H(s, \sqrt{-1}\Lambda_\omega[\phi, \phi^{*H}]s) \\ &\geq H(s, -\mathcal{K}_H(s)). \end{aligned}$$

Since $(E, \bar{\partial}_E, \phi)$ admits an approximate Hermitian-Einstein structure, it holds that for $\forall \xi > 0$, there exists a metric H_ξ such that

$$\sup_X |\mathcal{K}_{H_\xi} - \lambda \text{Id}| < \xi,$$

where $\lambda = \frac{2\pi \deg(E)}{\text{Vol}(X)\text{rank}(E)} < 0$. Taking $\xi = \frac{-\lambda}{2}$, there exists a Hermitian metric $H_{\frac{-\lambda}{2}}$ such that

$$(3.29) \quad \frac{3\lambda}{2} \cdot \text{Id} < \mathcal{K}_{H_{\frac{-\lambda}{2}}} < \frac{\lambda}{2} \cdot \text{Id}.$$

Combining (3.28) with (3.29), we have

$$(3.30) \quad -\frac{\lambda}{2} |s|^2 \leq \sqrt{-1}\Lambda_\omega \partial \bar{\partial} H_{\frac{-\lambda}{2}}(s, s).$$

Integrating both sides of (3.30) over X and using $\partial \bar{\partial} \omega^{n-1} = 0$, there holds

$$0 \leq \int_X |s|_{H_{\frac{-\lambda}{2}}}^2 \frac{\omega^n}{n!} \leq \int_X \sqrt{-1}\Lambda_\omega \partial \bar{\partial} H_{\frac{-\lambda}{2}}(s, s) = - \int_X \langle s, s \rangle_{H_{\frac{-\lambda}{2}}} \bar{\partial} \partial \frac{\omega^{n-1}}{(n-1)!} = 0.$$

This forces $s = 0$. □

Proof of Theorem 3.3 Let \mathcal{F} be any saturated Higgs sub-sheaf with rank p . Construct a Higgs bundle

$$\mathcal{G} = (G, \vartheta) = (\wedge^p E \otimes \det \mathcal{F}^{-1}, \vartheta),$$

where ϑ is the induced Higgs field. By using the technique in [17, p. 119], one can check that \mathcal{G} admits an approximate Hermitian-Einstein structure with the constant

$$(3.31) \quad \lambda(\mathcal{G}) = \frac{2p\pi}{\text{Vol}(X)} (\mu(E) - \mu(\mathcal{F})).$$

The canonical morphism $\det \mathcal{F} \hookrightarrow \wedge^p E$ induced by the inclusion map $i : \mathcal{F} \hookrightarrow E$ can be seen as a non-trivial ϑ -invariant section of \mathcal{G} . Then from Proposition 3.4, we have $\lambda(\mathcal{G}) = \frac{2\pi \deg(G)}{\text{Vol}(X)\text{rank}(G)} \geq 0$. This together with (3.31) indicates $\mu(\mathcal{F}) \leq \mu(E)$, i.e. $(E, \bar{\partial}_E, \phi)$ is semi-stable. □

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